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2003 J. Phys. A: Math. Gen. 36 9507

(http://iopscience.iop.org/0305-4470/36/36/306)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 9507–9532 PII: S0305-4470(03)59448-3

The third Painlevé equation and associated special polynomials

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Received 10 February 2003, in final form 14 July 2003
Published 27 August 2003
Online at stacks.iop.org/JPhysA/36/9507

Abstract

In this paper we are concerned with rational solutions, algebraic solutions and associated special polynomials with these solutions for the third Painlevé equation ($\mathrm{P_{III}}$). These rational and algebraic solutions of $\mathrm{P_{III}}$ are expressible in terms of special polynomials defined by second-order, bilinear differential-difference equations which are equivalent to Toda equations. The structure of the roots of these special polynomials is studied and it is shown that these have an intriguing, highly symmetric and regular structure. Using the Hamiltonian theory for $\mathrm{P_{III}}$, it is shown that these special polynomials satisfy pure difference equations, fourth-order, bilinear differential equations as well as differential-difference equations. Further, representations of the associated rational solutions in the form of determinants through Schur functions are given.

PACS number: 02.30.−f
Mathematics Subject Classification: 33E17, 34M35

1. Introduction

In this paper we are concerned with rational solutions and associated special polynomials for the third Painlevé equation ($\mathrm{P_{III}}$)

$$w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w} \tag{1.1}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$ and α, β, γ and δ are arbitrary constants. We remark that letting $w(z) = y(x)/\sqrt{x}$, with $x = \frac{1}{4}z^2$, in $\mathrm{P_{III}}$ yields

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\alpha y^2}{2x^2} + \frac{\beta}{2x} + \frac{\gamma y^3}{x^2} + \frac{\delta}{y} \tag{1.2}$$

which is known as $\mathrm{P_{III'}}$ (cf Okamoto [79]) and is often used to determine properties of solutions of $\mathrm{P_{III}}$. However (1.2) has algebraic solutions rather than rational solutions [68].

The six Painlevé equations (P_I–P_{VI}) were discovered by Painlevé, Gambier and their colleagues whilst studying second-order ordinary differential equations of the form

$$w'' = F(z, w, w') \tag{1.3}$$

where F is rational in w' and w and analytic in z. The Painlevé equations can be thought of as nonlinear analogues of the classical special functions. Indeed Iwasaki *et al* [45] characterize the six Painlevé equations as 'the most important nonlinear ordinary differential equations' and state that 'many specialists believe that during the twenty-first century the Painlevé functions will become new members of the community of special functions'. The general solutions of the Painlevé equations are transcendental in the sense that they cannot be expressed in terms of known elementary functions and so require the introduction of a new transcendental function to describe their solution.

Although first discovered from strictly mathematical considerations, the Painlevé equations have arisen in a variety of important physical applications, including statistical mechanics, plasma physics, nonlinear waves, quantum gravity, quantum field theory, general relativity, nonlinear optics and fibre optics. Further the Painlevé equations have attracted much interest since they arise in many physical situations and as reductions of the soliton equations which are solvable by inverse scattering (cf [1, 3], and references therein, for further details). Much of the current interest of the Painlevé equations is due to Wu, Tracy, McCoy and Barouch [94, 65], who showed that P_{III} appears in the theory of the Ising model, and Ablowitz and Segur [2], who demonstrated a close connection between completely integrable partial differential equations solvable by inverse scattering, the soliton equations, and the Painlevé equations.

It is well known that P_{II}–P_{VI} possess hierarchies of rational solutions for special values of the parameters (see, e.g., [6, 7, 11, 25, 30, 40, 41, 55, 61, 64, 66–68, 76–79, 90–93, 95, 97] and the references therein). These hierarchies are usually generated from 'seed solutions' using the associated Bäcklund transformations and frequently can be expressed in the form of determinants through 'τ-functions'.

Vorob'ev [92] and Yablonskii [95] expressed the rational solutions of the second Painlevé equation (P_{II})

$$w'' = 2w^3 + zw + \alpha \tag{1.4}$$

where α is an arbitrary constant, in terms of the logarithmic derivative of certain polynomials which are now known as the *Yablonskii–Vorob'ev polynomials*. Okamoto [78] obtained analogous polynomials related to some of the rational solutions of P_{IV}; these polynomials are now known as the *Okamoto polynomials*. Further Okamoto noted that they arise from special points in parameter space from the point of view of symmetry, which is associated with the affine Weyl group of type $A_2^{(2)}$. Umemura [89] associated analogous special polynomials with certain rational and algebraic solutions of P_{III}, P_V and P_{VI} which have similar properties to the Yablonskii–Vorob'ev polynomials and the Okamoto polynomials; see also [70, 87, 96]. Subsequently there have been several studies of special polynomials associated with the rational solutions of P_{II} [30, 48, 50, 83], the rational and algebraic solutions of P_{III} [49, 73], the rational solutions of P_{IV} [30, 51, 72], the rational solutions of P_V [63, 71] and the algebraic solutions of P_{VI} [53, 54, 62, 84, 85]. However the majority of these papers are concerned with the combinatorial structure and determinant representation of the polynomials, often related to the Hamiltonian structure and affine Weyl symmetries of the Painlevé equations. Typically these polynomials arise as the τ-functions for special solutions of the Painlevé equations and are generated through nonlinear, three-term recurrence relations which are Toda-type equations that arise from the associated Bäcklund transformations of the Painlevé equations. The coefficients of these special polynomials have some interesting, indeed somewhat mysterious, combinatorial properties (see [70, 87, 89]). Additionally these polynomials have been expressed as special cases of *Schur polynomials*, which are irreducible polynomial representations of the general linear group $\mathrm{GL}(n)$ and arise as τ-functions of the Kadomtsev–Petviashvili (KP) hierarchy [47]. The Yablonskii–Vorob'ev polynomials associated with $\mathrm{P_{II}}$ are expressible in terms of 2-reduced Schur functions [48, 50], and are related to the τ-function for the rational solution of the modified Korteweg-de Vries (mKdV) equation since $\mathrm{P_{II}}$ arises as a similarity reduction of the mKdV equation. Further, in [21], it is shown that the roots of these Yablonskii–Vorob'ev polynomials have a very symmetric, regular structure. The Okamoto polynomials associated with $\mathrm{P_{IV}}$ are expressible in terms of 3-reduced Schur functions [51, 72] since $\mathrm{P_{IV}}$ arises as a similarity reduction of the Boussinesq equation (cf [20]), which belongs to the so-called 3-reduction of the KP hierarchy [47].

It is also well known that $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ possess solutions which are expressible in terms of the classical special functions; these are often referred to as 'one-parameter families of solutions'. For $\mathrm{P_{II}}$ these special function solutions are expressed in terms of Airy functions $\mathrm{Ai}(z)$ [6, 24, 31, 78], for $\mathrm{P_{III}}$ in terms of Bessel functions $J_\nu(z)$ [58, 66, 68, 79], for $\mathrm{P_{IV}}$ in terms of Weber–Hermite (parabolic cylinder) functions $D_\nu(z)$ [11, 39, 57, 67, 78], for $\mathrm{P_V}$ in terms of Whittaker functions $M_{\kappa,\mu}(z)$, or equivalently confluent hypergeometric functions ${}_1F_1(a;c;z)$ [59, 36, 77, 93], and for $\mathrm{P_{VI}}$ in terms of hypergeometric functions ${}_2F_1(a,b;c;z)$ [27, 60, 76]; see also [1, 37, 40–42]. Some classical orthogonal polynomials, hereafter referred to as classical polynomials, arise as particular cases of these special function solutions and thus yield rational solutions of the associated Painlevé equations, especially in the representation of rational solutions through determinants. For $\mathrm{P_{III}}$ and $\mathrm{P_V}$ these are in terms of associated Laguerre polynomials $L_n^{(k)}(z)$ [17, 49, 63, 71], for $\mathrm{P_{IV}}$ in terms of Hermite polynomials $H_n(z)$ [11, 51, 67, 78] and for $\mathrm{P_{VI}}$ in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ [62, 85]. In fact all rational solutions of $\mathrm{P_{VI}}$ arise as particular cases of the special solutions given in terms of hypergeometric functions [64].

This paper is organized as follows. The special polynomials associated with rational solutions of $\mathrm{P_{III}}$, which occur in the generic case when $\gamma\delta \neq 0$, are studied in section 2. Using the Hamiltonian theory for $\mathrm{P_{III}}$, it is shown that these special polynomials also satisfy both differential equations and difference equations. Further these special polynomials are related to the determinantal form of rational solutions of $\mathrm{P_{III}}$. In section 3 we study the special polynomials associated with algebraic solutions of $\mathrm{P_{III}}$, which occur in the cases when either $\gamma = 0$ and $\alpha\delta \neq 0$, or $\delta = 0$ and $\beta\gamma \neq 0$. Again, using Hamiltonian theory, it is shown that these special polynomials also satisfy both differential equations and difference equations. Finally in section 4 we discuss our results and pose some open questions.

2. Rational solutions of $\mathrm{P_{III}}$

2.1. Introduction

In this section we consider the generic case of $\mathrm{P_{III}}$ when $\gamma\delta \neq 0$, then set $\gamma = 1$ and $\delta = -1$, without loss of generality (by rescaling w and z if necessary), and so consider

$$w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + w^3 - \frac{1}{w}. \tag{2.1}$$

The location of rational solutions for the generic case of $\mathrm{P_{III}}$ given by (2.1) is stated in the following theorem.

Theorem 2.1. *Equation (2.1), i.e.* $\mathrm{P_{III}}$ *with* $\gamma = -\delta = 1$, *has rational solutions if and only if* $\alpha + \varepsilon\beta = 4n$, *with* $n \in \mathbb{Z}$ *and* $\varepsilon = \pm 1$. *These rational solutions have the form*

$w = P_m(z)/Q_m(z)$, where $P_m(z)$ and $Q_m(z)$ are polynomials of degree m with no common roots.

Proof. See Gromak *et al* [41], p 174 (see also [66, 68, 91]). □

Hierarchies of rational solutions of the Painlevé equations can be obtained by applying Bäcklund transformations to 'seed solutions'. The Bäcklund transformations of P_{III}, which relate two solutions of P_{III} with different values of the parameters, are defined as follows. Suppose $w = w(z; \alpha, \beta, 1, -1)$ is a solution of P_{III}, then $w^{[j]} = w^{[j]}(z; \alpha^{[j]}, \beta^{[j]}, 1, -1)$, $j = 1, 2, \ldots, 6$, are also solutions of P_{III} where

$$\mathcal{T}^{[1]}: \qquad w^{[1]} = \frac{zw' + zw^2 - \beta w - w + z}{w(zw' + zw^2 + \alpha w + w + z)} \qquad \alpha^{[1]} = \alpha + 2 \qquad \beta^{[1]} = \beta + 2 \tag{2.2}$$

$$\mathcal{T}^{[2]}: \qquad w^{[2]} = -\frac{zw' - zw^2 - \beta w - w + z}{w(zw' - zw^2 - \alpha w + w + z)} \qquad \alpha^{[2]} = \alpha - 2 \qquad \beta^{[2]} = \beta + 2 \tag{2.3}$$

$$\mathcal{T}^{[3]}: \qquad w^{[3]} = -\frac{zw' + zw^2 + \beta w - w - z}{w(zw' + zw^2 + \alpha w + w - z)} \qquad \alpha^{[3]} = \alpha + 2, \quad \beta^{[3]} = \beta - 2 \tag{2.4}$$

$$\mathcal{T}^{[4]}: \qquad w^{[4]} = \frac{zw' - zw^2 + \beta w - w - z}{w(zw' - zw^2 - \alpha w + w - z)} \qquad \alpha^{[4]} = \alpha - 2, \quad \beta^{[4]} = \beta - 2 \tag{2.5}$$

$$\mathcal{T}^{[5]}: \qquad w^{[5]} = -w \qquad \alpha^{[5]} = -\alpha \qquad \beta^{[5]} = -\beta \tag{2.6}$$

$$\mathcal{T}^{[6]}: \qquad w^{[6]} = 1/w \qquad \alpha^{[6]} = -\beta \qquad \beta^{[6]} = -\alpha \tag{2.7}$$

[34, 35] (see also [66, 68, 91] and the references therein).

We remark that the rational solutions of the generic case of P_{III} (2.1) lie on the lines $\alpha + \varepsilon\beta = 4n$ in the α–β plane, rather than isolated points as is the case for P_{IV}. Thus the Bäcklund transformations (2.3) and (2.4) map a rational solutions to itself. Further, equation (2.1) is of type D_6 in the terminology of Sakai [81], who studied the Painlevé equations through a geometric approach based on rational surfaces.

2.2. Associated special polynomials

Umemura [89], see also [49, 70, 87], derived special polynomials associated with rational solutions of P_{III}, which are defined in theorem 2.2; though as explained below these are actually polynomials in $1/z$ rather than z. Further Umemura states that these 'polynomials' are the analogues of the Yablonskii–Vorob'ev polynomials associated with rational solutions of P_{II} and the Okamoto polynomials associated with rational solutions of P_{IV}.

Theorem 2.2. *Suppose that $T_n(z; \mu)$ satisfies the recursion relation*

$$zT_{n+1}T_{n-1} = -z\left[T_n\frac{d^2T_n}{dz^2} - \left(\frac{dT_n}{dz}\right)^2\right] - T_n\frac{dT_n}{dz} + (z + \mu)T_n^2 \tag{2.8}$$

Table 1. Polynomials $T_n(1/\xi;\mu)$ associated with rational solutions of P_{III} due to Umemura [89].

$$T_1(1/\xi;\mu) = 1+\mu\xi$$
$$T_2(1/\xi;\mu) = 1+3\mu\xi+3\mu^2\xi^2+\mu(\mu^2-1)\xi^3$$
$$T_3(1/\xi;\mu) = 1+6\mu\xi+15\mu^2\xi^2+5\mu(4\mu^2-1)\xi^3+15\mu^2(\mu^2-1)\xi^4+ 3\mu(\mu^2-1)(2\mu^2-3)\xi^5+\mu^2(\mu^2-1)(\mu^2-4)\xi^6$$
$$T_4(1/\xi;\mu) = 1+10\mu\xi+45\mu^2\xi^2+15\mu(8\mu^2-1)\xi^3+105\mu^2(2\mu^2-1)\xi^4+ 63\mu(\mu^2-1)(4\mu^2-1)\xi^5+105\mu^2(\mu^2-1)(2\mu^2-3)\xi^6+ 15\mu(\mu^2-1)(8\mu^4-27\mu^2+15)\xi^7+45\mu^2(\mu^2-1)(\mu^2-2)(\mu^2-4)\xi^8+ 5\mu^3(\mu^2-1)(\mu^2-4)(2\mu^2-11)\xi^9+\mu^2(\mu^2-1)^2(\mu^2-4)(\mu^2-9)\xi^{10}$$

Table 2. Rational solutions of P_{III} arising from the polynomials in table 1.

$$w_0(z;\mu)=1$$
$$w_1(z;\mu)=1-\frac{1}{z+\mu}$$
$$w_2(z;\mu)=1+\frac{1}{z+\mu-1}-\frac{3(z+\mu)^2}{(z+\mu)^3-\mu}$$
$$w_3(z;\mu)=1+\frac{3(z+\mu-1)^2}{(z+\mu-1)^3-\mu+1}-\frac{6(z+\mu)^5-15\mu(z+\mu)^2+9\mu}{(z+\mu)^6-5\mu(z+\mu)^3+9\mu(z+\mu)-5\mu^2}$$

with $T_{-1}(z;\mu)=1$ *and* $T_0(z;\mu)=1$. *Then*

$$\begin{aligned} w_n(z;\mu) \equiv w(z;\alpha_n,\beta_n,1,-1) &= 1+\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left[\frac{T_{n-1}(z;\mu-1)}{z^n T_n(z;\mu)}\right]\right\} \\ &= \frac{T_n(z;\mu-1)T_{n-1}(z;\mu)}{T_n(z;\mu)T_{n-1}(z;\mu-1)} \end{aligned} \tag{2.9}$$

satisfies P_{III}, *with* $\alpha_n=2n+2\mu-1$ *and* $\beta_n=2n-2\mu+1$.

Remark 2.3

(i) The first few polynomials $T_n(1/\xi;\mu)$, where $z=1/\xi$, for P_{III} defined by (2.8) are given in table 1 and associated rational solutions of P_{III} are given in table 2.

(ii) It is clear from the recurrence relation (2.8) that the $T_n(z;\mu)$ are rational functions, though it is not obvious that in fact they are polynomials in $\xi=1/z$, since one is dividing by $T_{n-1}(z;\mu)$ at every iteration. Indeed it is somewhat remarkable that $T_n(1/\xi;\mu)$ defined by (2.8) are polynomials in ξ.

(iii) The recurrence relation (2.8) for $T_n(z;\mu)$ can be rewritten in the form

$$\left[\frac{z}{2}\mathrm{D}_z^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}-(z+\mu)\right]T_n\bullet T_n = -zT_{n+1}T_{n-1} \tag{2.10}$$

where D_z is the Hirota operator defined by

$$\mathrm{D}_z F(z)\bullet G(z) = \left[\left(\frac{\mathrm{d}}{\mathrm{d}z_1}-\frac{\mathrm{d}}{\mathrm{d}z_2}\right)F(z_1)G(z_2)\right]_{z_1=z_2=z}. \tag{2.11}$$

(iv) Making the transformation

$$T_n(z)=\exp\left(\tfrac{1}{4}z^2+\mu z+\tfrac{1}{2}n^2\ln z\right)\tau_n(z)$$

in (2.8) yields the Toda equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n\right)+\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}=0. \tag{2.12}$$

(v) The hierarchy of rational solutions of P_{III} given in table 2 can also be derived using the Bäcklund transformation $\mathcal{T}^{[1]}$ (2.2), i.e.

$$w_{n+1} = \frac{zw_n' + zw_n^2 - 2(n-\mu+1)w_n + z}{w_n\left[zw_n' + zw_n^2 + 2(n+\mu)w_n + z\right]}$$
$$\alpha_n = 2n + 2\mu - 1 \qquad \beta_n = 2n - 2\mu + 1 \tag{2.13}$$

where $w_m \equiv w(z; \alpha_m, \beta_m, 1, -1)$, with 'seed solution'

$$w_0(z; \alpha_0, \beta_0; 1; -1) = 1 \qquad \alpha_0 = 2\mu - 1 \qquad \beta_0 = -2\mu + 1.$$

The inverse transformation, derived from the Bäcklund transformation $\mathcal{T}^{[4]}$ (2.5), is

$$w_{n-1} = \frac{zw_n' - zw_n^2 + 2(n-\mu)w_n - z}{w_n\left[zw_n' - zw_n^2 - 2(n+\mu-1)w_n - z\right]}. \tag{2.14}$$

Hence eliminating w_n' between (2.13) and (2.14) yields the difference equation

$$\frac{2n+1}{w_n w_{n+1} - 1} + \frac{2n-1}{w_n w_{n-1} - 1} + zw_n + 2n + 2\mu + \frac{z}{w_n} = 0. \tag{2.15}$$

Setting $w_n = \mathrm{i}v_n$ and $z = \mathrm{i}x$ yields

$$\frac{2n+1}{v_n v_{n+1} + 1} + \frac{2n-1}{v_n v_{n-1} + 1} + xv_n - 2(n+\mu) - \frac{x}{v_n} = 0 \tag{2.16}$$

which is an alternative $\mathrm{dP_{II}}$ [22, 26, 69].

(vi) The rational solution $w_n(z)$ has the form $w_n = P_{n^2}(z)/Q_{n^2}(z)$, where $P_{n^2}(z)$ and $Q_{n^2}(z)$ are polynomials of degree n^2 with no common roots.

The 'polynomials' $T_n(z;\mu)$ are somewhat unsatisfactory since they are polynomials in $\xi = 1/z$ rather than polynomials in z, which would be more natural and is the case for the Yablonskii–Vorob'ev polynomials and Okamoto polynomials associated with rational solutions of P_{II} and P_{IV}, respectively. Umemura [89] makes the transformation $T_n(z;\mu) = \widetilde{T}_n(\xi;\mu)$, with $z = 1/\xi$. Then $\widetilde{T}_n(\xi;\mu)$ are polynomials in ξ and satisfy the differential-difference equation

$$\widetilde{T}_{n+1}\widetilde{T}_{n-1} + \xi^4\left[\widetilde{T}_n\frac{\mathrm{d}^2\widetilde{T}_n}{\mathrm{d}\xi^2} - \left(\frac{\mathrm{d}\widetilde{T}_n}{\mathrm{d}\xi}\right)^2\right] + \xi^3\widetilde{T}_n\frac{\mathrm{d}\widetilde{T}_n}{\mathrm{d}\xi} - (1+\mu\xi)\widetilde{T}_n^2 = 0$$

with $\widetilde{T}_0 = 1$ and $\widetilde{T}_1 = 0$, though this approach requires that a transformation is made to P_{III}. However it is straightforward to determine a sequence of functions $S_n(z;\mu)$, which are generated through an equation, that are polynomials in z and also do not require that P_{III} is transformed. These are given in the following theorem.

Theorem 2.4. *Suppose that $S_n(z;\mu)$ satisfies the recursion relation*

$$S_{n+1}S_{n-1} = -z\left[S_n\frac{\mathrm{d}^2S_n}{\mathrm{d}z^2} - \left(\frac{\mathrm{d}S_n}{\mathrm{d}z}\right)^2\right] - S_n\frac{\mathrm{d}S_n}{\mathrm{d}z} + (z+\mu)S_n^2 \tag{2.17}$$

with $S_{-1}(z;\mu) = S_0(z;\mu) = 1$. Then

$$w_n = w(z;\alpha_n,\beta_n,1,-1) = 1 + \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left[\frac{S_{n-1}(z;\mu-1)}{S_n(z;\mu)}\right]\right\}$$
$$\equiv \frac{S_n(z;\mu-1)S_{n-1}(z;\mu)}{S_n(z;\mu)S_{n-1}(z;\mu-1)} \tag{2.18}$$

Table 3. Polynomials associated with rational solutions of P_{III}.

$S_1(z;\mu) = z+\mu$
$S_2(z;\mu) = (z+\mu)^3 - \mu$
$S_3(z;\mu) = (z+\mu)^6 - 5\mu(z+\mu)^3 + 9\mu(z+\mu) - 5\mu^2$
$S_4(z;\mu) = (z+\mu)^{10} - 15\mu(z+\mu)^7 + 63\mu(z+\mu)^5 - 225\mu(z+\mu)^3 + 315\mu^2(z+\mu)^2 - 175\mu^3(z+\mu) + 36\mu^2$
$S_5(z;\mu) = (z+\mu)^{15} - 35\mu(z+\mu)^{12} + 252\mu(z+\mu)^{10} + 175\mu^2(z+\mu)^9 - 2025\mu(z+\mu)^8 + 945\mu^2(z+\mu)^7 - 1225\mu(\mu^2-9)(z+\mu)^6 - 26082\mu^2(z+\mu)^5 + 33075\mu^3(z+\mu)^4 - 350\mu^2(35\mu^2+36)(z+\mu)^3 + 11340\mu^3(z+\mu)^2 - 225\mu^2(49\mu^2-36)(z+\mu) + 7\mu^3(875\mu^2-828)$

satisfies P_{III} *with* $\alpha_n = 2n+2\mu-1$ *and* $\beta_n = 2n-2\mu+1$ *and*

$$\widehat{w}_n = w(z;\widehat{\alpha}_n,\widehat{\beta}_n,1,-1) = 1 + \frac{d}{dz}\left\{\ln\left[\frac{S_{n-1}(z;\mu)}{S_n(z;\mu-1)}\right]\right\} \equiv \frac{S_n(z;\mu)S_{n-1}(z;\mu-1)}{S_n(z;\mu-1)S_{n-1}(z;\mu)} \tag{2.19}$$

satisfies P_{III} *with* $\widehat{\alpha}_n = -2n+2\mu-1$ *and* $\widehat{\beta}_n = -2n-2\mu+1$.

Proof. This result essentially follows from theorem 1 due to Kajiwara and Masuda [49] since equation (2.17), modulo a scaling factor, is equation (16) in proposition 3 of [49]; see also remark 2.7. Further, note that $\widehat{w}_n = 1/w_n$, which is a consequence of the Bäcklund transformation (2.7). However, we believe that the polynomials $S_n(z;\mu)$ have not been written down previously.

The first few polynomials $S_n(z;\mu)$, which are monic polynomials of degree $\frac{1}{2}n(n+1)$, are given in table 3. The associated rational solutions of P_{III} are given in table 2. The rational solutions of P_{III} defined by (2.18) and (2.19) can be generalized using the Bäcklund transformation (2.6) to include all those described in theorem 2.1 satisfying the condition $\alpha+\beta=4n$. Rational solutions of P_{III} satisfying the condition $\alpha-\beta=4n$ are obtained by letting $w \to \mathrm{i}w$ and $z \to \mathrm{i}z$ in (2.18) and (2.19), and then using the Bäcklund transformation (2.6). Thus

$$w_n^* = w(z;\alpha_n^*,\beta_n^*,1,-1) = \mathrm{i} + \frac{d}{dz}\left\{\ln\left[\frac{S_{n-1}(\mathrm{i}z;\mu-1)}{S_n(\mathrm{i}z;\mu)}\right]\right\} \equiv \mathrm{i}\frac{S_n(\mathrm{i}z;\mu-1)S_{n-1}(\mathrm{i}z;\mu)}{S_n(\mathrm{i}z;\mu)S_{n-1}(\mathrm{i}z;\mu-1)} \tag{2.20}$$

satisfies P_{III} with $\alpha_n^* = 2\mu+2n-1$ and $\beta_n^* = 2\mu-2n-1$ and

$$\widehat{w}_n^* = w(z;\widehat{\alpha}_n^*,\widehat{\beta}_n^*,1,-1) = \mathrm{i} + \frac{d}{dz}\left\{\ln\left[\frac{S_{n-1}(\mathrm{i}z;\mu)}{S_n(\mathrm{i}z;\mu-1)}\right]\right\} \equiv \mathrm{i}\frac{S_n(\mathrm{i}z;\mu)S_{n-1}(\mathrm{i}z;\mu-1)}{S_n(\mathrm{i}z;\mu-1)S_{n-1}(\mathrm{i}z;\mu)} \tag{2.21}$$

satisfies P_{III} with $\widehat{\alpha}_n^* = 2\mu-2n-1$ and $\widehat{\beta}_n^* = 2\mu+2n-1$. We note that $\widehat{w}_n = -1/w_n$, due to the Bäcklund transformations (2.6) and (2.7).

In figures 1–3 plots of the roots of the polynomials $S_3(\xi-\mu,\mu)$, $S_4(\xi-\mu,\mu)$ and $S_5(\xi-\mu,\mu)$ defined by (2.3) for various μ are given, respectively. Initially for $\mu=-2$, $\mu=-3$ and $\mu=-3.5$, respectively, there is an approximate triangle with 3, 4

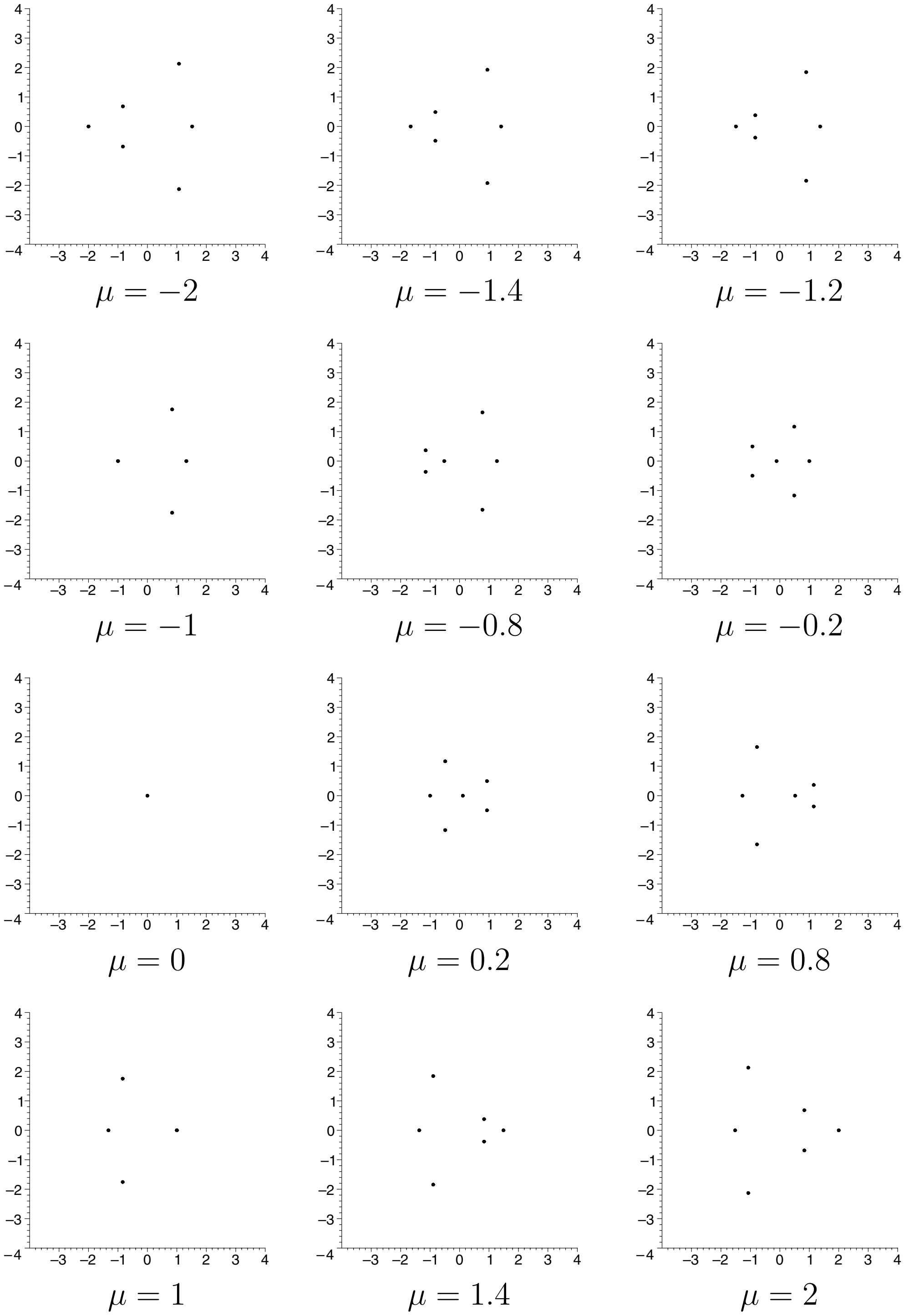

Figure 1. Roots of the polynomial $S_3(\xi - \mu, \mu)$ for various μ.

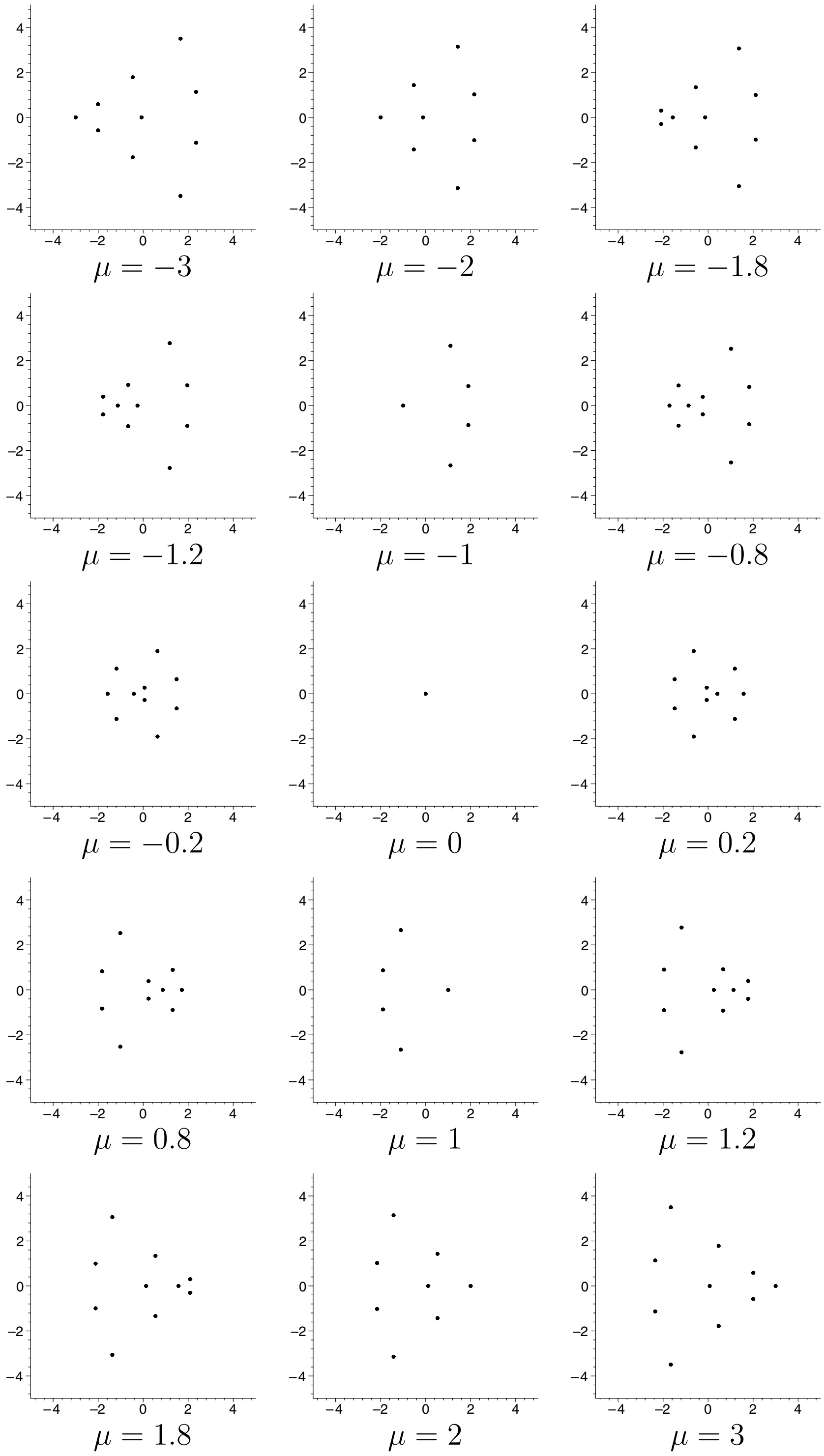

Figure 2. Roots of the polynomial $S_4(\xi - \mu, \mu)$ for various μ.

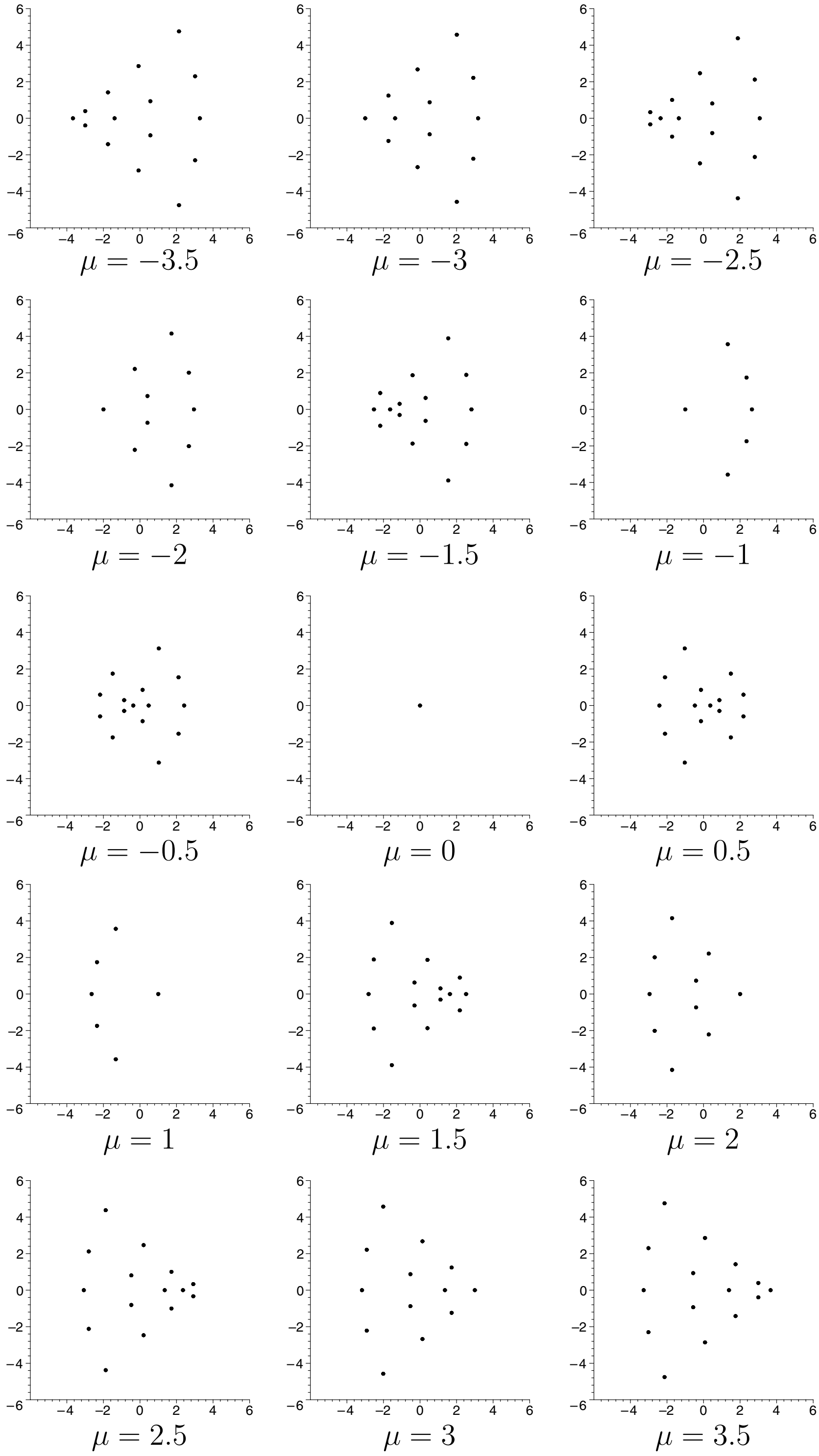

Figure 3. Roots of the polynomial $S_5(\xi - \mu, \mu)$ for various μ.

and 5 roots, respectively, on each side. Then as μ increases, the roots then in turn coalesce and eventually give for $\mu = 2$, $\mu = 3$ and $\mu = 3.5$, respectively, another approximate triangle with its orientation reversed—see remark 2.5(vi). □

Remark 2.5

(i) The polynomials $S_n(z;\mu)$ defined by (2.17) are related to $T_n(z;\mu)$ defined by (2.8) through $S_n(z;\mu) = z^{n(n+1)/2}T_n(z;\mu)$. In view of this, it is a little surprising that $T_n(z;\mu)$, rather than $S_n(z;\mu)$, appear in [49, 70, 87, 89].

(ii) The polynomials $S_n(z;\mu)$ have the property that $S_n(z;\mu) = S_n(-z;-\mu)$.

(iii) It is clear from the recurrence relation (2.17) that the $S_n(z;\mu)$ are rational functions, though it is not obvious that in fact they are polynomials since one is dividing by $S_{n-1}(z;\mu)$ at every iteration. Indeed it is somewhat remarkable that the $S_n(z;\mu)$ defined by (2.17) are polynomials.

(iv) The recurrence relation (2.17) for the polynomials $S_n(z;\mu)$ can be rewritten in the form

$$\left[\frac{z}{2}\mathrm{D}_z^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z} - (z+\mu)\right] S_n \bullet S_n = -S_{n+1}S_{n-1} \tag{2.22}$$

where D_z is the Hirota operator defined by (2.11).

(v) Making the transformation $S_n(z;\mu) = \exp\left(\frac{1}{4}z^2 + \mu z\right)\tau_n(z)$ in (2.8) yields the Toda equation (2.12).

(vi) It is straightforward to determine when the roots of $S_3(z;\mu)$–$S_5(z;\mu)$ coalesce using discriminants of polynomials. Let $f(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$ be a monic polynomial of degree m with roots $\alpha_1, \alpha_2, \ldots, \alpha_m$, so $f(z) = \prod_{j=1}^m (z-\alpha_j)$. Then the discriminant of f is

$$\mathrm{Dis}(f) = \prod_{1\leqslant j<k\leqslant m} (\alpha_j - \alpha_k)^2.$$

Hence the polynomial f has a multiple root when $\mathrm{Dis}(f) = 0$. It is straightforward to show that

$$\begin{aligned}
\mathrm{Dis}(S_3(z;\mu)) &= 3^{12}5^5\mu^6(\mu^2-1)^2\\
\mathrm{Dis}(S_4(z;\mu)) &= 3^{27}5^{20}7^7\mu^{14}(\mu^2-1)^6(\mu^2-4)^2\\
\mathrm{Dis}(S_5(z;\mu)) &= 3^{66}5^{45}7^{28}\mu^{26}(\mu^2-1)^{14}(\mu^2-4)^6(\mu^2-9)^2.
\end{aligned}$$

Thus $S_3(z;\mu)$ has multiple roots when $\mu = 0, \pm 1$ (at $z = 0$), $S_4(z;\mu)$ when $\mu = 0, \pm 1, \pm 2$ (at $z = 0$), and $S_5(z;\mu)$ when $\mu = 0, \pm 1, \pm 2, \pm 3$ (at $z = 0$). These are the values of μ for which the roots of $S_3(z;\mu)$–$S_5(z;\mu)$ coalesce in figures 1–3, respectively.

2.3. Hamiltonian theory for $\mathrm{P_{III}}$

The Hamiltonian associated with $\mathrm{P_{III}}$ is [75, 78] (see also [28])

$$\mathrm{H_{III}} = p^2q^2 - zpq^2 - (\beta-1)pq + zp + \tfrac{1}{2}(\beta - 2 - \alpha)zq \tag{2.23}$$

and so from Hamilton's equations

$$z\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathrm{H_{III}}}{\partial p} \qquad z\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathrm{H_{III}}}{\partial p} \tag{2.24}$$

we obtain the system

$$\begin{aligned}
z\frac{\mathrm{d}q}{\mathrm{d}z} &= 2pq^2 - zq^2 - (\beta-1)q + z\\
z\frac{\mathrm{d}p}{\mathrm{d}z} &= -2p^2q + 2zpq + (\beta-1)p - \frac{1}{2}(\beta-2-\alpha)z.
\end{aligned} \tag{2.25}$$

Setting $q = w$ and eliminating p in this system yields P_{III} (2.1). Setting $p(z) = \sqrt{x}/[1-y(x)]$, with $x = z^2$, and eliminating q yields P_V

$$\frac{d^2y}{dx^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + \frac{(y-1)^2}{x^2}\left(ay + \frac{b}{y}\right) + \frac{cy}{x} + \frac{dy(y+1)}{y-1} \tag{2.26}$$

with

$$a = (\alpha - \beta + 2)^2/32 \qquad b = -(\alpha + \beta - 2)^2/32 \qquad c = -\tfrac{1}{2} \qquad d = 0.$$

It is well known that P_V (2.26) with $d = 0$ is equivalent to P_{III} [35, 41].

Next, following Jimbo and Miwa [46] and Okamoto [75, 78], we define the auxiliary Hamiltonian function σ by

$$\sigma = \tfrac{1}{2}H_{III} + \tfrac{1}{2}pq + \tfrac{1}{8}(\beta - 2)^2 - \tfrac{1}{4}z^2 \tag{2.27}$$

where p and q satisfy the Hamiltonian system (2.25). Then σ satisfies the second-order, second-degree equation given by

$$(z\sigma'' - \sigma')^2 + 4(\sigma')^2(z\sigma' - 2\sigma) + 4z\lambda_1\sigma' - z^2(z\sigma' - 2\sigma + 2\lambda_0) = 0 \tag{2.28}$$

with $\lambda_1 = -\frac{1}{4}\alpha(\beta - 2)$ and $\lambda_0 = \frac{1}{8}\alpha^2 + \frac{1}{8}(\beta - 2)^2$, which is sometimes referred to as the 'Jimbo–Miwa–Okamoto σ-equation'. Conversely if σ is a solution of (2.28) then the solution of the system (2.25) is

$$q = \frac{2z\sigma'' + 2(1-\beta)\sigma' - \alpha z}{z^2 - 4(\sigma')^2} \qquad p = \sigma' + \frac{1}{2}z. \tag{2.29}$$

Due to the relationship between the Hamiltonian and the τ-function (see [78]), it can be shown that solutions of (2.28) have the form

$$\sigma(z) = z\frac{d}{dz}\ln\left\{z^{1/8}\exp\left(\frac{1}{8}z^2\right)\tau_n(z)\right\} = \frac{1}{4}z^2 + \frac{1}{8} + z\frac{d}{dz}\ln\tau_n(z)$$

where τ_n satisfies the Toda equation (2.12). Hence, since $\tau_n(z) = \exp\left(-\frac{1}{4}z^2 - \mu z\right)S_n(z;\mu)$, then rational solutions of (2.28) have the form

$$\sigma_{n,\mu}(z) = -\frac{1}{4}z^2 - \mu z + \frac{1}{8} + z\frac{d}{dz}\ln S_n(z;\mu) \tag{2.30}$$

with $\lambda_1 = \mu^2 - \left(n + \frac{1}{2}\right)^2$ and $\lambda_0 = \mu^2 + \left(n + \frac{1}{2}\right)^2$. Note that w_n, the rational solution of P_{III} defined by (2.19), is related to the auxiliary Hamiltonian function $\sigma_{n,\mu}$ through

$$w_n = (\sigma_{n-1,\mu-1} - \sigma_{n,\mu})/z. \tag{2.31}$$

Furthermore, using proposition 4.8 in [28] (who discuss the Hamiltonian for $P_{III'}$ rather than P_{III}), it can be shown that $\sigma_{n,\mu}$ defined by (2.30) also satisfies the following two third-order difference equations

$$\begin{aligned} z^2 &+ [\sigma_{n+1,\mu} - \sigma_{n,\mu} - (n+1+\mu)][\sigma_{n+1,\mu} - \sigma_{n,\mu} - (n+1-\mu)] \\ &\times \frac{(\sigma_{n+1,\mu} - \sigma_{n-1,\mu})(\sigma_{n+2,\mu} - \sigma_{n,\mu})}{(\sigma_{n+1,\mu} - \sigma_{n-1,\mu} - 2n - 1)(\sigma_{n+2,\mu} - \sigma_{n,\mu} - 2n - 3)} = 0 \end{aligned} \tag{2.32}$$

which is a difference equation in n, and

$$\begin{aligned} (n+\mu+1)(n-\mu)z^2 &+ (\sigma_{n,\mu+1} - \sigma_{n,\mu-1})(\sigma_{n,\mu+2} - \sigma_{n,\mu}) \\ &\times \left[(\mu+1)\sigma_{n,\mu} - \mu\sigma_{n,\mu+1} + \tfrac{1}{4}z^2 + \tfrac{1}{2}\mu(\mu+1) - \tfrac{1}{2}\left(n + \tfrac{1}{2}\right)^2\right] = 0 \end{aligned} \tag{2.33}$$

which is a difference equation in μ.

Multiplying (2.28) by $1/z^2$ and the differentiating with respect to z yields

$$z^2\sigma''' - z\sigma'' + 6z(\sigma')^2 - 8\sigma\sigma' + \sigma' - \tfrac{1}{2}z^3 + 2z\lambda_1 = 0. \tag{2.34}$$

Then substituting (2.30) and $\lambda_1 = \mu^2 - \left(n + \frac{1}{2}\right)^2$ into this yields the fourth-order, bilinear equation for S_n

$$\begin{aligned} &z^2[S_n S_n'''' - 4S_n' S_n''' + 3(S_n'')^2] + 2z(S_n S_n''' - S_n' S_n'') \\ &\qquad - 4z(z+\mu)[S_n S_n'' - (S_n')^2] - 2S_n S_n'' + 4\mu S_n S_n' = 2n(n+1)S_n^2. \end{aligned} \tag{2.35}$$

We remark that substituting (2.30) into (2.28), yields a third-order, quad-linear equation for S_n, which is considerably more complex than (2.35). Hence S_n satisfies the differential equation (2.35) as well as the differential-difference equation (2.17).

Now we shall derive a pure difference equation for S_n. Consider the functions p_n and q_n, which satisfy the Hamiltonian system (2.25) with $\alpha = 2n + 2\mu - 1$ and $\beta = 2n - 2\mu + 1$, i.e.

$$\begin{aligned} z\frac{\mathrm{d}q_n}{\mathrm{d}z} &= 2p_n q_n^2 - zq_n^2 - 2(n-\mu)q_n + z \\ z\frac{\mathrm{d}p_n}{\mathrm{d}z} &= -2p_n^2 q_n + 2zp_n q_n + 2(n-\mu)p_n + 2\mu z. \end{aligned} \tag{2.36}$$

In terms of the auxiliary Hamiltonian function $\sigma_{n,\mu}$ defined by (2.30), then using (2.29) and (2.31), it follows that q_n and p_n are given by

$$q_n = (\sigma_{n-1,\mu-1} - \sigma_{n,\mu})/z \qquad p_n = \frac{1}{2}z + \frac{\mathrm{d}\sigma_{n-1,\mu}}{\mathrm{d}z} \tag{2.37}$$

and hence from (2.17)

$$p_n = \frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}\ln S_{n-1}(z;\mu)\right) - \mu \equiv z - \frac{S_n(z;\mu)S_{n-2}(z;\mu)}{S_{n-1}^2(z;\mu)}. \tag{2.38}$$

Further, using equations (4.40)–(4.43) in the proof of proposition 4.6 in [28] (where the Hamiltonian for $\mathrm{P_{III'}}$ rather than $\mathrm{P_{III}}$ is discussed), it can be shown that q_n and p_n satisfy the discrete system

$$q_{n+1} = \frac{1}{q_n} - \frac{2n+1}{q_n^2 p_n + 2\mu q_n + z} \tag{2.39}$$

$$p_{n+1} = -q_n^2 p_n - 2\mu q_n \tag{2.40}$$

$$q_{n-1} = \frac{p_n - z}{q_n p_n - zq_n - 2n + 1} \tag{2.41}$$

$$p_{n-1} = -q_n^2 p_n - 2\mu q_n + (2q_n p_n - 2n + 2\mu + 1)\frac{2n-1}{p_n - z} - z\left(\frac{2n-1}{p_n - z}\right)^2. \tag{2.42}$$

Solving (2.39) or (2.41) for p_n and then substituting it into (2.40) or (2.42) yields the second-order difference equation (2.15), which is equivalent to the alternative $\mathrm{dP_{II}}$ (2.16), with $w_n = q_n$. A difference equation for p_n can be obtained as follows. Subtracting (2.42) from (2.40) yields

$$p_{n+1} - p_{n-1} = -(2n-1)\frac{2q_n p_n - 2n + 2\mu + 1}{p_n - z} + z\left(\frac{2n-1}{p_n - z}\right)^2 \tag{2.43}$$

and then solving for q_n yields

$$q_n = -\frac{(p_{n+1} - p_{n-1})(p_n - z)^2 + 2(2n-1)\mu(p_n - z) - (2n-1)^2 p_n}{2(2n-1)p_n(p_n - z)}. \tag{2.44}$$

By substituting (2.38) into (2.44), then we can express q_n in terms of the polynomials $S_n = S_n(z;\mu)$

$$q_n = \frac{\left(S_{n+1}S_{n-2}^2 - S_n^2 S_{n-3}\right)S_{n-1}}{2(2n-1)S_nS_{n-2}\left(S_nS_{n-2} - zS_{n-1}^2\right)} - \frac{(2n-1-2\mu)S_{n-1}^2}{2\left(S_nS_{n-2} - zS_{n-1}^2\right)} + \frac{(2n-1)zS_{n-1}^4}{2S_nS_{n-2}\left(S_nS_{n-2} - zS_{n-1}^2\right)}. \tag{2.45}$$

Since $q_n = w_n$, the solution of $\mathrm{P_{III}}$ for $\alpha = 2n+2\mu-1$, $\beta = 2n-2\mu+1$, $\gamma = 1$ and $\delta = -1$, then substituting (2.45) into the difference equation (2.15) yields a sixth-order, hexa-linear difference equation for S_n, which is omitted due to its size as it has 67 operands.

We remark that this difference equation for S_n can also be obtained by first substituting (2.44) into (2.39), which yields the third-order difference equation for p_n

$$\frac{(p_{n+2} - p_n)(p_{n+1} - z)^2 + 2(2n+1)\mu(p_{n+1} - z) + (2n+1)^2 p_{n+1}}{2(2n+1)p_{n+1}(p_{n+1} - z)} = \frac{2(2n-1)p_n(p_n - z)}{(p_{n+1} - p_{n-1})(p_n - z)^2 + 2(2n-1)\mu(p_n - z) - (2n-1)^2 p_n}. \tag{2.46}$$

Then substituting $p_n = z - S_nS_{n-2}\big/S_{n-1}^2$ into (2.38) yields a sixth-order, hexa-linear difference equation for S_n.

We remark that there are solutions of the discrete system (2.39) with $\mu = 0$ given by

$$q_n = \mathrm{i}u_{n+1}/u_n \qquad p_n = \mathrm{i}xu_n^2 \qquad z = \mathrm{i}x \tag{2.47}$$

where u_n is a solution of the special case of $\mathrm{dP_{II}}$ [33, 80]

$$u_{n+1} + u_{n-1} = \frac{(2n+1)u_n}{x\left(1 - u_n^2\right)}. \tag{2.48}$$

The relationship between τ-functions for $\mathrm{P_{III'}}$ and $\mathrm{dP_{II}}$ (2.48) is discussed in [5, 9, 13, 29].

2.4. Determinantal form of rational solutions of $\mathrm{P_{III}}$

Kajiwara and Masuda [49] derived representations of rational solutions for $\mathrm{P_{III}}$ in the form of determinants, which are described in the following theorem.

Theorem 2.6. *Let $p_k(z;\mu)$ be the polynomial defined by*

$$\sum_{k=0}^{\infty} p_k(z;\mu)\lambda^k = (1+\lambda)^\mu \exp(z\lambda) \tag{2.49}$$

with $p_k(z;\mu) = 0$ for $k < 0$, and $\tau_n(z)$, for $n \geqslant 1$, be the $n \times n$ determinant

$$\tau_n(z;\mu) = \begin{vmatrix} p_1(z;\mu) & p_3(z;\mu) & \cdots & p_{2n-1}(z;\mu) \\ p_0(z;\mu) & p_2(z;\mu) & \cdots & p_{2n-2}(z;\mu) \\ \vdots & \vdots & \ddots & \vdots \\ p_{-n+2}(z;\mu) & p_{-n+4}(z;\mu) & \cdots & p_n(z;\mu) \end{vmatrix}. \tag{2.50}$$

Then

$$w_n(z) = 1 + \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left[\frac{\tau_{n-1}(z;\mu-1)}{\tau_n(z;\mu)}\right]\right\} = \frac{\tau_n(z;\mu-1)\tau_{n-1}(z;\mu)}{\tau_n(z;\mu)\tau_{n-1}(z;\mu-1)} \tag{2.51}$$

for $n \geqslant 1$, satisfies $\mathrm{P_{III}}$ *with* $(\alpha_n, \beta_n, \gamma_n, \delta_n) = (2n+2\mu-1, 2n-2\mu+1, 1, -1)$.

Remark 2.7

(i) Note that $p_k(z;\mu) = L_k^{(\mu-k)}(-z)$, where $L_k^{(m)}(\zeta)$ is the associated Laguerre polynomial (cf [4, 8, 86]), which are orthogonal polynomials on the interval $0 \leqslant \zeta \leqslant \infty$, with respect to the weight function $\zeta^m \exp(-\zeta)$, and are also defined by

$$L_k^{(m)}(\zeta) = \frac{\zeta^{-m}\,\mathrm{e}^{\zeta}}{k!}\frac{\mathrm{d}^k}{\mathrm{d}\zeta^k}(\mathrm{e}^{-\zeta}\zeta^{m+k}) \qquad k > -1.$$

(ii) The function $\tau_n(z;\mu)$ defined by (2.50) can also be written as

$$\tau_n(z;\mu) = \mathcal{W}(p_1(z;\mu), p_3(z;\mu), \ldots, p_{2n-1}(z;\mu)) \tag{2.52}$$

where $\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is the Wronskian defined by

$$\mathcal{W}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \begin{vmatrix} \varphi_1(z) & \varphi_2(z) & \cdots & \varphi_n(z) \\ \varphi_1'(z) & \varphi_2'(z) & \cdots & \varphi_n'(z) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)}(z) & \varphi_2^{(n-1)}(z) & \cdots & \varphi_n^{(n-1)}(z) \end{vmatrix} \tag{2.53}$$

since $\frac{\partial p_m}{\partial z}(z;\mu) = p_{m-1}(z;\mu)$, which is immediate from (2.49).

(iii) The function $\tau_n(z;\mu)$ defined by (2.50) satisfies the equation

$$(2n+1)\tau_{n+1}\tau_{n-1} = -z\left[\tau_n\frac{\mathrm{d}^2\tau_n}{\mathrm{d}z^2} - \left(\frac{\mathrm{d}\tau_n}{\mathrm{d}z}\right)^2\right] - \tau_n\frac{\mathrm{d}\tau_n}{\mathrm{d}z} + (z+\mu)\tau_n^2 \tag{2.54}$$

which is equation (16) in proposition 3 of [49]. Further it is straightforward, by comparing (2.17) and (2.54), to show that

$$\tau_n(z;\mu) = c_n S_n(z;\mu) \qquad c_n = \prod_{j=1}^{n}(2j+1)^{j-n}. \tag{2.55}$$

3. Algebraic solutions of $\mathrm{P_{III}}$

3.1. Introduction

In this section we consider the special case of $\mathrm{P_{III}}$ when either (i) $\gamma = 0$ and $\alpha\delta \neq 0$, or (ii) $\delta = 0$ and $\beta\gamma \neq 0$. In case (i), we make the transformation

$$w(z) = \left(\tfrac{2}{3}\right)^{1/2} u(\zeta) \qquad z = \left(\tfrac{2}{3}\right)^{3/2}\zeta^3 \tag{3.1}$$

and set $\alpha = 1$, $\beta = 2\mu$ and $\delta = -1$, with μ an arbitrary constant, without loss of generality, which yields

$$\frac{\mathrm{d}^2u}{\mathrm{d}\zeta^2} = \frac{1}{u}\left(\frac{\mathrm{d}u}{\mathrm{d}\zeta}\right)^2 - \frac{1}{\zeta}\frac{\mathrm{d}u}{\mathrm{d}\zeta} + 4\zeta u^2 + 12\mu\zeta - \frac{4\zeta^4}{u}. \tag{3.2}$$

In case (ii), we make the transformation

$$w(z) = \left(\tfrac{3}{2}\right)^{1/2}\big/u(\zeta) \qquad z = \left(\tfrac{2}{3}\right)^{3/2}\zeta^3 \tag{3.3}$$

and set $\alpha = 2\mu$, $\beta = -1$ and $\gamma = 1$, with μ an arbitrary constant, without loss of generality, which again yields (3.2). The scalings in (3.1) and (3.3) have been chosen so that the associated special polynomials are monic polynomials. We remark that equation (3.2) is of type D_7 in the terminology of Sakai [81]; we shall refer to it as $\mathrm{P_{III}^{(7)}}$. Studies of properties of solutions of (3.2) include [34, 38, 41, 56, 58, 66, 68, 73, 74].

Rational solutions of (3.2) correspond to algebraic solutions of P_{III} with $\gamma = 0$ and $\alpha\delta \neq 0$, or $\delta = 0$ and $\beta\gamma \neq 0$. Lukashevich [56, 58] obtained algebraic solutions of P_{III}, which are classified in the following theorem.

Theorem 3.1. *Equation (3.2) has rational solutions if and only if $\mu = n$, with $n \in \mathbb{Z}$. These rational solutions have the form $u(\zeta) = P_{n^2+1}(\zeta)/Q_{n^2}(\zeta)$, where $P_{n^2+1}(\zeta)$ and $Q_{n^2}(\zeta)$ are monic polynomials of degree $n^2 + 1$ and n^2, respectively.*

Proof. See Gromak *et al* [41], p 164 (see also [34, 38, 66, 68]). □

A straightforward method for generating rational solutions of (3.2) is through the Bäcklund transformation

$$u_{\mu+\varepsilon} = \frac{\zeta^3}{u_\mu^2} + \frac{\varepsilon\zeta}{2u_\mu^2}\frac{\mathrm{d}u_\mu}{\mathrm{d}\zeta} - \frac{3(2\mu+\varepsilon)}{2u_\mu} \tag{3.4}$$

where $\varepsilon^2 = 1$ and u_μ is the solution of (3.2) for parameter μ, using the 'seed solution' $u_0(\zeta) = \zeta$ for $\mu = 0$ (see Gromak *et al* [41], p 164—see also [34, 38, 66, 68]). Further we note that $u_{-\mu}(\zeta) = -\mathrm{i}u_\mu(\mathrm{i}\zeta)$. Therefore the transformation group for (3.2) is isomorphic to the affine Weyl group $\widetilde{A}_1$, which also is the transformation group for P_{II} [78, 88, 90].

3.2. Associated special polynomials

Ohyama [73] derived special polynomials associated with the rational solutions of (3.2). These are essentially described in theorem 3.2, though here the variables have been scaled and the expression of the rational solutions of (3.2) in terms of these special polynomials is explicitly given.

Theorem 3.2. *Suppose that $R_n(\zeta)$ satisfies the recursion relation*

$$2\zeta R_{n+1}R_{n-1} = -R_n\frac{\mathrm{d}^2R_n}{\mathrm{d}\zeta^2} + \left(\frac{\mathrm{d}R_n}{\mathrm{d}\zeta}\right)^2 - \frac{R_n}{\zeta}\frac{\mathrm{d}R_n}{\mathrm{d}\zeta} + 2(\zeta^2 - n)R_n^2 \tag{3.5}$$

with $R_0(\zeta) = 1$ and $R_1(\zeta) = \zeta^2$. Then

$$u_n(\zeta) = \frac{R_{n+1}(\zeta)R_{n-1}(\zeta)}{R_n^2(\zeta)} \equiv \frac{\zeta^2 - n}{\zeta} - \frac{1}{2\zeta^2}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left\{\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln R_n(\zeta)\right\} \tag{3.6}$$

satisfies (3.2) with $\mu = n$. Additionally $u_{-n}(\zeta) = -\mathrm{i}u_n(\mathrm{i}\zeta)$.

Remark 3.3

(i) The first few polynomials $R_n(\zeta)$ defined by (3.5) are given in table 4 and associated rational solutions of (3.2) are given in table 5.

(ii) The polynomial $R_n(\zeta)$ is a monic polynomial of degree $\frac{1}{2}n(n+3)$ with integer coefficients. Further it has the form $R_n(\zeta) = V_n(\zeta)\zeta^{\kappa_n}$, with $\kappa_n = \frac{1}{2}n^2 - \frac{1}{4}[1 - (-1)^n]$, where $V_n(\zeta)$ is a monic polynomial of degree $\frac{3}{4}n + \frac{1}{8}[1 - (-1)^n]$ with simple zeros and $V_n(0) \neq 0$. These polynomials appear to be analogous to the Yablonskii–Vorob'ev polynomials for P_{II} and it is an open problem whether they can be expressed as Schur polynomials as is the case for the Yablonskii–Vorob'ev polynomials [48, 50]. The polynomials $V_n(\zeta)$ are generated by the recurrence relation

$$V_n\frac{\mathrm{d}^2V_n}{\mathrm{d}\zeta^2} - \left(\frac{\mathrm{d}V_n}{\mathrm{d}\zeta}\right)^2 + \frac{V_n}{\zeta}\frac{\mathrm{d}V_n}{\mathrm{d}\zeta} - 2(\zeta^2 - n)V_n^2 = \begin{cases} -2\zeta^2V_{n+1}V_{n-1} & \text{if } n \text{ even} \\ -2V_{n+1}V_{n-1} & \text{if } n \text{ odd} \end{cases} \tag{3.7}$$

with $V_0 = 1$ and $V_1 = 1$ (see theorem 3.3 in [73]).

Table 4. Polynomials generated by (3.5) which are associated with rational solutions of $P_{III}^{(7)}$ (3.2).

$R_2(\zeta) = (\zeta^2 - 1)\zeta^3$
$R_3(\zeta) = (\zeta^4 - 4\zeta^2 + 5)\zeta^5$
$R_4(\zeta) = (\zeta^8 - 10\zeta^6 + 40\zeta^4 - 70\zeta^2 + 35)\zeta^6$
$R_5(\zeta) = (\zeta^{12} - 20\zeta^{10} + 175\zeta^8 - 840\zeta^6 + 2275\zeta^4 - 3220\zeta^2 + 1925)\zeta^8$
$R_6(\zeta) = (\zeta^{18} - 35\zeta^{16} + 560\zeta^{14} - 5320\zeta^{12} - 32690\zeta^{10} + 133070\zeta^8 - 354200\zeta^6 + 585200\zeta^4 - 525525\zeta^2 + 175175)\zeta^9$
$R_7(\zeta) = (\zeta^{24} - 56\zeta^{22} + 1470\zeta^{20} - 23800\zeta^{18} + 263375\zeta^{16} - 2088240\zeta^{14} + 12105940\zeta^{12} - 51466800\zeta^{10} + 158533375\zeta^8 - 343343000\zeta^6 + 493643150\zeta^4 - 421821400\zeta^2 + 163788625)\zeta^{11}$
$R_8(\zeta) = (\zeta^{32} - 84\zeta^{30} + 3360\zeta^{28} - 84700\zeta^{26} + 1501500\zeta^{24} - 19787460\zeta^{22} + 199916640\zeta^{20} - 1574673100\zeta^{18} + 9741481750\zeta^{16} - 47328781500\zeta^{14} + 179306327200\zeta^{12} - 521782561300\zeta^{10} + 1136861225500\zeta^8 - 1778744467500\zeta^6 + 1860638780000\zeta^4 - 1132762130500\zeta^2 + 283190532625)\zeta^{12}$

Table 5. Rational solutions of $P_{III}^{(7)}$ (3.2) arising from the polynomials in table 4.

$u_1(\zeta) = \dfrac{\zeta^2 - 1}{\zeta}$
$u_2(\zeta) = \dfrac{\zeta(\zeta^4 - 4\zeta^2 + 5)}{(\zeta^2 - 1)^2}$
$u_3(\zeta) = \dfrac{(\zeta^2 - 1)(\zeta^8 - 10\zeta^6 + 40\zeta^4 - 70\zeta^2 + 35)}{\zeta(\zeta^4 - 4\zeta^2 + 5)^2}$
$u_4(\zeta) = \dfrac{\zeta(\zeta^4 - 4\zeta^2 + 5)(\zeta^{12} - 20\zeta^{10} + 175\zeta^8 - 840\zeta^6 + 2275\zeta^4 - 3220\zeta^2 + 1925)}{(\zeta^8 - 10\zeta^6 + 40\zeta^4 - 70\zeta^2 + 35)^2}$

(iii) Making the transformation

$$R_n(\zeta) = \zeta^{-n(n+1)} \exp\left(\tfrac{1}{8}\zeta^8 - \tfrac{1}{2}n\zeta^2\right) \tau_n(\zeta) \tag{3.8}$$

in (3.5) yields the Toda equation

$$\frac{d}{d\zeta}\left(\zeta \frac{d}{d\zeta} \ln \tau_n\right) + 2\frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2} = 0. \tag{3.9}$$

(iv) From equation (3.4) we have

$$\begin{aligned} u_{n+1} &= \frac{\zeta^3}{u_n^2} + \frac{\zeta}{2u_n^2}\frac{du_n}{d\zeta} - \frac{3(2n+1)}{2u_n} \\ u_{n-1} &= \frac{\zeta^3}{u_n^2} - \frac{\zeta}{2u_n^2}\frac{du_n}{d\zeta} - \frac{3(2n-1)}{2u_n}. \end{aligned}$$

Hence eliminating $du_n/d\zeta$ yields the difference equation for u_n

$$u_{n+1} + u_{n-1} = \frac{2\zeta^3}{u_n^2} - \frac{6n}{u_n} \tag{3.10}$$

which is an alternative discrete P_I'' [26, 32] (see also [22, 43]). Thus substituting (3.6), we see that $R_n(\zeta)$ satisfies the fifth-order, tri-linear difference equation

$$R_{n+2}R_{n-1}^2 + R_{n-2}R_{n+1}^2 = 2\zeta^3 R_n^3 - 6n R_{n+1} R_n R_{n-1}. \tag{3.11}$$

Table 6. Polynomials generated by (3.13).

$U_2(\zeta) = \zeta^2 - 1$
$U_3(\zeta) = (\zeta^4 - 4\zeta^2 + 5)\zeta$
$U_4(\zeta) = \zeta^8 - 10\zeta^6 + 40\zeta^4 - 70\zeta^2 + 35$
$U_5(\zeta) = (\zeta^{12} - 20\zeta^{10} + 175\zeta^8 - 840\zeta^6 + 2275\zeta^4 - 3220\zeta^2 + 1925)\zeta$
$U_6(\zeta) = \zeta^{18} - 35\zeta^{16} + 560\zeta^{14} - 5320\zeta^{12} - 32690\zeta^{10} + 133070\zeta^8 - 354200\zeta^6 + 585200\zeta^4 - 525525\zeta^2 + 175\,175$
$U_7(\zeta) = (\zeta^{24} - 56\zeta^{22} + 1470\zeta^{20} - 23800\zeta^{18} + 263375\zeta^{16} - 2088240\zeta^{14} + 12105940\zeta^{12} - 51466800\zeta^{10} + 158533375\zeta^8 - 343343000\zeta^6 + 493643150\zeta^4 - 421821400\zeta^2 + 163788625)\zeta$
$U_8(\zeta) = \zeta^{32} - 84\zeta^{30} + 3360\zeta^{28} - 84700\zeta^{26} + 1501500\zeta^{24} - 19787460\zeta^{22} + 199916640\zeta^{20} - 1574673100\zeta^{18} + 9741481750\zeta^{16} - 47328781500\zeta^{14} + 179306327200\zeta^{12} - 521782561300\zeta^{10} + 1136861225500\zeta^8 - 1778744467500\zeta^6 + 1860638780000\zeta^4 - 1132762130500\zeta^2 + 283\,190\,532\,625$

To discuss the locations of the poles of the rational solutions of (3.2), we define the polynomials $U_n(\zeta)$ by

$$\begin{aligned} U_{2n}(\zeta) &= V_{2n}(\zeta) = \zeta^{-3n} R_{2n}(\zeta) \\ U_{2n+1}(\zeta) &= \zeta V_{2n+1}(\zeta) = \zeta^{-3n-1} R_{2n+1}(\zeta). \end{aligned} \tag{3.12}$$

It is routine to show that the polynomials $U_n(\zeta)$ are generated by the recurrence relation

$$U_n \frac{\mathrm{d}^2 U_n}{\mathrm{d}\zeta^2} - \left(\frac{\mathrm{d}U_n}{\mathrm{d}\zeta}\right)^2 + \frac{U_n}{\zeta}\frac{\mathrm{d}U_n}{\mathrm{d}\zeta} - 2(\zeta^2 - n)U_n^2 = \begin{cases} -2U_{n+1}U_{n-1} & \text{if } n \text{ even} \\ -2\zeta^2 U_{n+1}U_{n-1} & \text{if } n \text{ odd} \end{cases} \tag{3.13}$$

with $U_0 = 1$ and $U_1 = \zeta$. The first few polynomials $U_n(\zeta)$ are given in table 6.

In figure 4 plots of the locations of the poles of the algebraic solutions of $\mathrm{P}_{\mathrm{III}}^{(7)}$ (3.2) given by $u_n(\zeta)$, for $n = 3, 4, \ldots, 8$, as defined in (5), which are equivalent to the locations of the roots of $U_n(\zeta)$, are given. These plots show that the locations of the poles have a very symmetric, regular structure and take the form of two 'triangles' in a 'bow-tie' shape. For the algebraic solution $u_{2n}(\zeta)$, with $n \geqslant 1$, the poles in the 'triangles' are in arcs with $1, 3, \ldots, 2n+1$ poles, whilst for $u_{2n+1}(\zeta)$, with $n \geqslant 1$, the poles in the 'triangles' are in arcs with $2, 4, \ldots, 2n+2$ poles together with a pole at the origin. These plots are invariant under reflections in the real and imaginary axes and the poles lie in the sectors $-\frac{1}{6}\pi < \arg(\zeta) < \frac{1}{6}\pi$ and $\frac{5}{6}\pi < \arg(\zeta) < \frac{7}{6}\pi$.

3.3. Hamiltonian theory for $\mathrm{P}_{\mathrm{III}}^{(7)}$

A Hamiltonian associated with $\mathrm{P}_{\mathrm{III}}^{(7)}$ (3.2) is

$$\mathrm{H}_{\mathrm{III}}^{(7)}(p, q; \kappa) = p^2 q^2 + 6\left(\kappa - \tfrac{1}{2}\right)pq - 2\zeta^3(p + q) \tag{3.14}$$

which is obtained by transforming the Hamiltonian in [73, 81], and so from Hamilton's equations

$$\zeta\frac{\mathrm{d}q}{\mathrm{d}\zeta} = \frac{\partial \mathrm{H}_{\mathrm{III}}^{(7)}}{\partial p} \qquad \zeta\frac{\mathrm{d}p}{\mathrm{d}\zeta} = -\frac{\partial \mathrm{H}_{\mathrm{III}}^{(7)}}{\partial p} \tag{3.15}$$

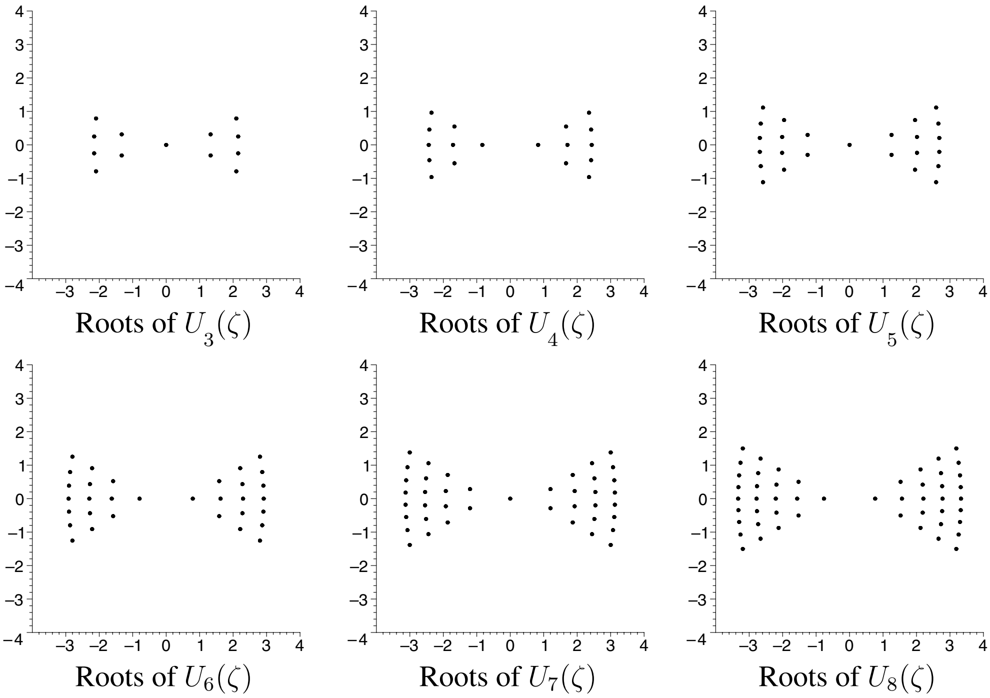

Figure 4. Poles of algebraic solutions of $\mathrm{P}_{\mathrm{III}}^{(7)}$ (3.2).

we obtain the system

$$\begin{aligned} \zeta \frac{\mathrm{d}q}{\mathrm{d}\zeta} &= 2pq^2 + 6\left(\kappa - \frac{1}{2}\right) q - 2\zeta^3 \\ \zeta \frac{\mathrm{d}p}{\mathrm{d}\zeta} &= -2p^2 q - 6\left(\kappa - \frac{1}{2}\right) p + 2\zeta^3. \end{aligned} \tag{3.16}$$

Setting $p = u$ and eliminating q in this system yields $\mathrm{P}_{\mathrm{III}}^{(7)}$ (3.2) with $\mu = \kappa$, whilst setting $q = u$ and eliminating p yields (3.2) with $\mu = \kappa - 1$, and so $p = u_\mu$ and $q = u_{\mu-1}$. Now define the auxiliary Hamiltonian function

$$\sigma = \tfrac{1}{6}\mathrm{H}_{\mathrm{III}}^{(7)}(p, q; \mu) + \tfrac{1}{2}pq + \tfrac{3}{2}\mu^2 = \tfrac{1}{6}p^2q^2 - \tfrac{1}{3}(p+q)\zeta^3 + \mu pq + \tfrac{3}{2}\mu^2 \tag{3.17}$$

where p and q satisfy (3.16). Then σ satisfies the second-order, second-degree equation

$$\left(\zeta \frac{\mathrm{d}^2\sigma}{\mathrm{d}\zeta^2} - 5\frac{\mathrm{d}\sigma}{\mathrm{d}\zeta}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\zeta}\right)^2 \left(\zeta \frac{\mathrm{d}\sigma}{\mathrm{d}\zeta} - 6\sigma\right) - 48\mu\zeta^5 \frac{\mathrm{d}\sigma}{\mathrm{d}\zeta} = 16\zeta^{10}. \tag{3.18}$$

Conversely, if σ is a solution of (3.18), then solutions of (3.16) are given by

$$p = -\frac{1}{2\zeta^2}\frac{\mathrm{d}\sigma}{\mathrm{d}\zeta} \qquad q = \zeta^2 \left[\zeta \frac{\mathrm{d}^2\sigma}{\mathrm{d}\zeta^2} + (6\mu - 5)\frac{\mathrm{d}\sigma}{\mathrm{d}\zeta} + 4\zeta^5\right] \Bigg/ \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\zeta}\right)^2.$$

Since $p = u_\mu$ and $q = u_{\mu-1}$, where u_μ satisfies (3.2), then rational solutions of the Hamiltonian system (3.16) with $\kappa = n$ have the form

$$p_n(\zeta) = \frac{R_{n+1}(\zeta)R_{n-1}(\zeta)}{R_n^2(\zeta)} \qquad q_n(\zeta) = p_{n-1}(\zeta) = \frac{R_n(\zeta)R_{n-2}(\zeta)}{R_{n-1}^2(\zeta)}. \tag{3.19}$$

It is straightforward to show, using the relationship between solutions of (3.16) and (3.18) together with (3.5), that rational solutions of (3.18) with $\mu = n$ have the form

$$\begin{aligned}\sigma_n &= \frac{1}{6}p_n^2q_n^2 - \frac{1}{3}(p_n+q_n)\zeta^3 + np_nq_n + \frac{3}{2}n^2\\ &= -\frac{1}{2}\zeta^4 + n\zeta^2 - \frac{3}{2}n + \frac{1}{6} + \zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln R_n.\end{aligned} \tag{3.20}$$

We remark that from (3.8) we obtain

$$\sigma_n = -n^2 - \frac{5}{2}n + \frac{1}{6} + \zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln\tau_n.$$

Also from (3.10) and (3.19) it follows that (p_n, q_n) satisfy the discrete system

$$p_{n+1} = \frac{2\zeta^3}{p_n^2} - \frac{6n}{p_n} - q_n \qquad q_{n+1} = p_n.$$

Dividing (3.18) by ζ^{10}, setting $\mu = n$ and then differentiating with respect to ζ yields the third-order equation

$$\zeta^2\frac{\mathrm{d}^3\sigma_n}{\mathrm{d}\zeta^3} - 9\zeta\frac{\mathrm{d}^2\sigma_n}{\mathrm{d}\zeta^2} + 6\zeta\left(\frac{\mathrm{d}\sigma_n}{\mathrm{d}\zeta}\right)^2 + (25-24\sigma_n)\frac{\mathrm{d}\sigma_n}{\mathrm{d}\zeta} = 24n\zeta^5. \tag{3.21}$$

Substituting (3.20) into this equation yields the fourth-order, bilinear equation for R_n

$$\begin{aligned}&\zeta^3\left[R_n\frac{\mathrm{d}^4R_n}{\mathrm{d}\zeta^4} - 4\frac{\mathrm{d}R_n}{\mathrm{d}\zeta}\frac{\mathrm{d}^3R_n}{\mathrm{d}\zeta^3} + 3\left(\frac{\mathrm{d}^2R_n}{\mathrm{d}\zeta^2}\right)^2\right] - 6\zeta^2\left(R_n\frac{\mathrm{d}^3R_n}{\mathrm{d}\zeta^3} - \frac{\mathrm{d}R_n}{\mathrm{d}\zeta}\frac{\mathrm{d}^2R_n}{\mathrm{d}\zeta^2}\right)\\ &\quad - 12\zeta(\zeta^4-3n-1)\left[R_n\frac{\mathrm{d}^2R_n}{\mathrm{d}\zeta^2} - \left(\frac{\mathrm{d}R_n}{\mathrm{d}\zeta}\right)^2\right] - 9\zeta\left[R_n\frac{\mathrm{d}^2R_n}{\mathrm{d}\zeta^2} + \left(\frac{\mathrm{d}R_n}{\mathrm{d}\zeta}\right)^2\right]\\ &\quad + 3(12\zeta^4 - 16n\zeta^2 + 12n + 7)R_n\frac{\mathrm{d}R_n}{\mathrm{d}\zeta} - 24n\zeta[(n+3)\zeta^2 - 3n - 1]R_n^2 = 0.\end{aligned} \tag{3.22}$$

We remark that substituting (3.20) into (3.18), yields a third-order, quad-linear equation for R_n. Therefore the polynomials $R_n(\zeta)$ satisfy the fourth-order, bilinear equation differential equation (3.22), the fifth-order, tri-linear difference equation (3.11), as well as the bilinear differential-difference equation (3.5). It is straightforward to show that R_n satisfies additional differential-difference equations. From (3.20) and (3.19), then it follows that R_n also satisfies the quad-linear differential-difference equation

$$\begin{aligned}\zeta R_nR_{n-1}^2\frac{\mathrm{d}R_n}{\mathrm{d}\zeta} &= \frac{1}{6}R_{n+1}^2R_{n-2}^2 - \frac{1}{3}\zeta^3\left(R_{n+1}R_{n-1}^3 + R_n^3R_{n-2}\right) + nR_{n+1}R_nR_{n-1}R_{n-2}\\ &\quad + \left[\frac{1}{2}\zeta^4 - n\zeta^2 + \frac{3}{2}n(n+1) - \frac{1}{6}\right]R_n^2R_{n-1}^2.\end{aligned}$$

Further, we remark that substituting (3.19) into (3.16) and adding the two resulting equations yields the tri-linear differential-difference equation

$$\zeta R_n R_{n-1}\frac{\mathrm{d}R_{n+1}}{\mathrm{d}\zeta} - 2\zeta R_{n+1}R_{n-1}\frac{\mathrm{d}R_n}{\mathrm{d}\zeta} + \zeta R_n R_{n+1}\frac{\mathrm{d}R_{n-1}}{\mathrm{d}\zeta} = R_{n+2}R_{n-1}^2 - R_{n-2}R_{n+1}^2 + 3R_{n+1}R_nR_{n-1}$$

whilst subtracting them yields the difference equation (3.11).

4. Conclusions

In this paper we have studied properties of special polynomials associated with rational and algebraic solutions of $\mathrm{P_{III}}$. In particular we have demonstrated that the roots of these polynomials have a very symmetric, regular structure. These are analogous to the results in [19, 21], where it is shown that the roots of the polynomials associated with rational solutions of $\mathrm{P_{II}}$ and $\mathrm{P_{IV}}$ also have a very symmetric, regular structure. This seems to be yet another remarkable property of the Painlevé equations, indeed more generally of 'integrable' differential equations.

Bracken *et al* [14] show that multivortex solutions of the complex sine-Gordon and sinh-Gordon equations on the complex plane associated with a Weierstrass-type system given by

$$\frac{\partial^2 u}{\partial z\partial\bar{z}} - \frac{\sigma\bar{u}}{1+\sigma|u|^2}\frac{\partial u}{\partial z}\frac{\partial u}{\partial\bar{z}} + \frac{1}{4}u(1+\sigma|u|^2) = 0 \qquad \sigma = \pm 1 \tag{4.1}$$

can be expressed in terms of $T_n(z;\mu)$ defined in section 2. We remark that equation (4.1) was derived in the context of the reduction of the $O(4)$ nonlinear sigma model and as well the reduction of the self-dual Yang–Mills equations and relativistic equations (cf [10, 12, 82]).

An important, well-known property of classical polynomials, such as the Hermite, Laguerre or Legendre polynomials whose roots all lie on the real line (cf [4, 8, 86]), is that the roots of successive polynomials interlace. For a set of classical polynomials $\varphi_n(z)$, for $n = 0, 1, 2, \ldots$, if $z_{n,m}$ and $z_{n,m+1}$ are two successive roots of $\varphi_n(z)$, i.e. $\varphi_n(z_{n,m}) = 0$ and $\varphi_n(z_{n,m+1}) = 0$, then $\varphi_{n-1}(\zeta_{n-1}) = 0$ and $\varphi_{n+1}(\zeta_{n+1}) = 0$ for some ζ_{n-1} and ζ_{n+1} such that $z_{n,m} < \zeta_{n-1}, \zeta_{n+1} < z_{n,m+1}$. Further the derivatives $\varphi_n'(z)$ and $\varphi_{n+1}'(z)$ also have roots in the interval $(z_{n,m}, z_{n,m+1})$, i.e. $\varphi_n'(\xi_n) = 0$ and $\varphi_{n+1}'(\xi_{n+1}) = 0$ for some ξ_n and ξ_{n+1} such that $z_{n,m} < \xi_n, \xi_{n+1} < z_{n,m+1}$.

An interesting open question is whether there are analogous results for the polynomials $S_n(z;\mu)$ and $R_n(\zeta)$. Obviously there are significant differences since the polynomials $S_n(z;\mu)$ and $R_n(\zeta)$ have $\frac{1}{2}n(n+1)$ and $\frac{1}{2}n(n+3)$ complex roots, respectively, whereas the classical polynomial $\varphi_n(z)$ has real roots. The pattern of the roots of $S_n(z;\mu)$ and $R_n(\zeta)$ is highly symmetric and structured, suggesting that they have interesting properties. A particularly intriguing question is whether there is any 'interlacing of roots' (in the complex plane), analogous to that for classical polynomials (on the real line); though we do not expect any specific relationship between the roots of the polynomials $S_n(z;\mu)$ and $R_n(\zeta)$ with roots of any classical polynomial.

To investigate a possible 'interlacing of roots', in figure 5 the roots of two successive polynomials $R_n(\zeta)$, denoted by $\bullet$, and $R_{n+1}(\zeta)$, denoted by $\circ$, are plotted for $n = 4, 5, \ldots, 9$ and in figure 6 the roots of $R_n(\zeta)$, denoted by $\bullet$, and $R_{n+1}'(\zeta)$, denoted by $\diamond$, are plotted for $n = 4, 5, \ldots, 9$. These figures suggest that there is some structure to the relative positions of the roots. In particular, in figure 5 the roots of $R_n(\zeta)$ appear to lie within triangles formed by joining the nearest neighbours of the roots of $R_{n+1}(\zeta)$, at least for small values of n.

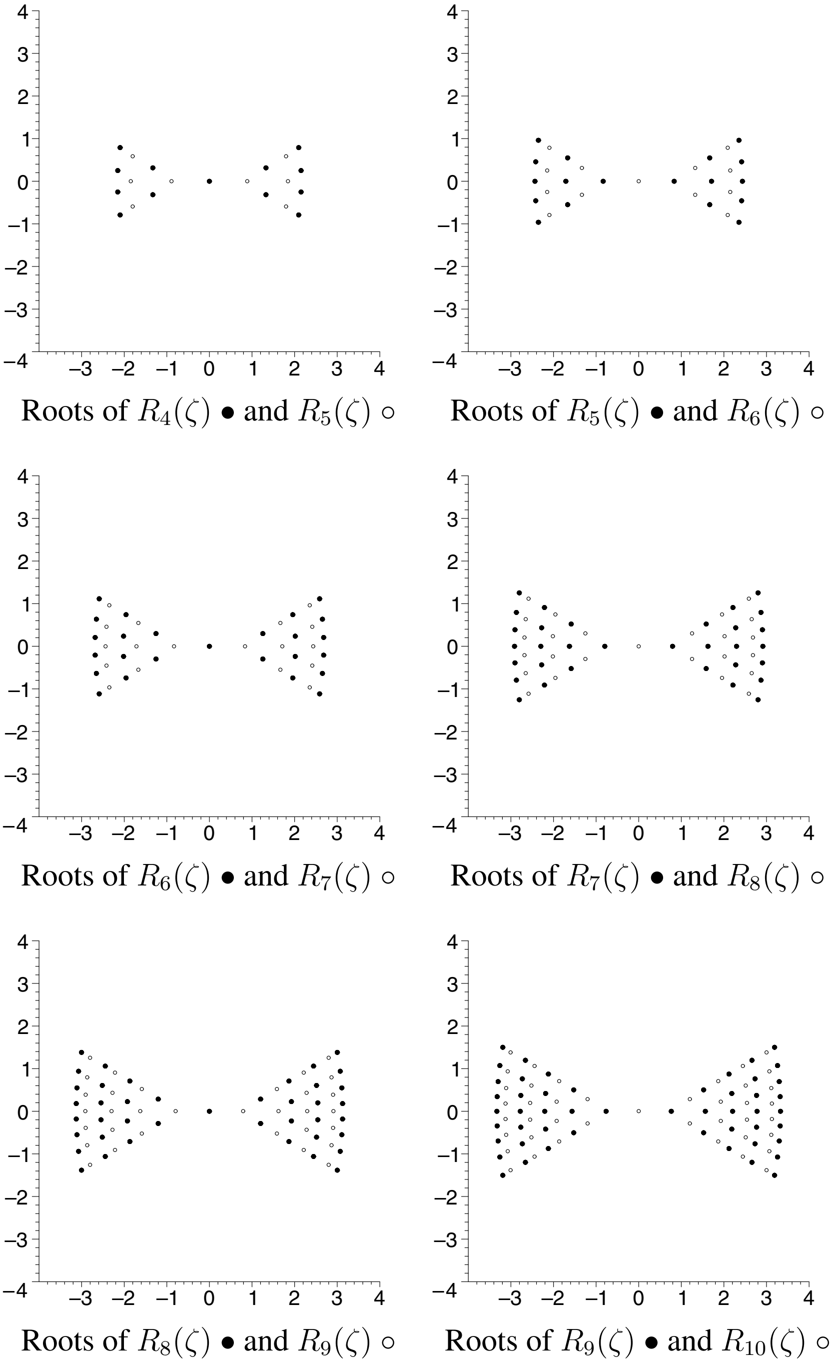

Figure 5. Roots of $R_n(\zeta)$, denoted by $\bullet$, and $R_{n+1}(\zeta)$, denoted by $\circ$, for $n = 4, 5, \ldots, 9$.

Analogously, in figure 6 the roots of $R_n(\zeta)$ appear to lie within triangles formed by joining the nearest neighbours of the roots of $R'_{n+1}(\zeta)$, again for small values of n. A similar structure is observed in [21] for the roots of the Yablonskii–Vorob'ev polynomials which are associated

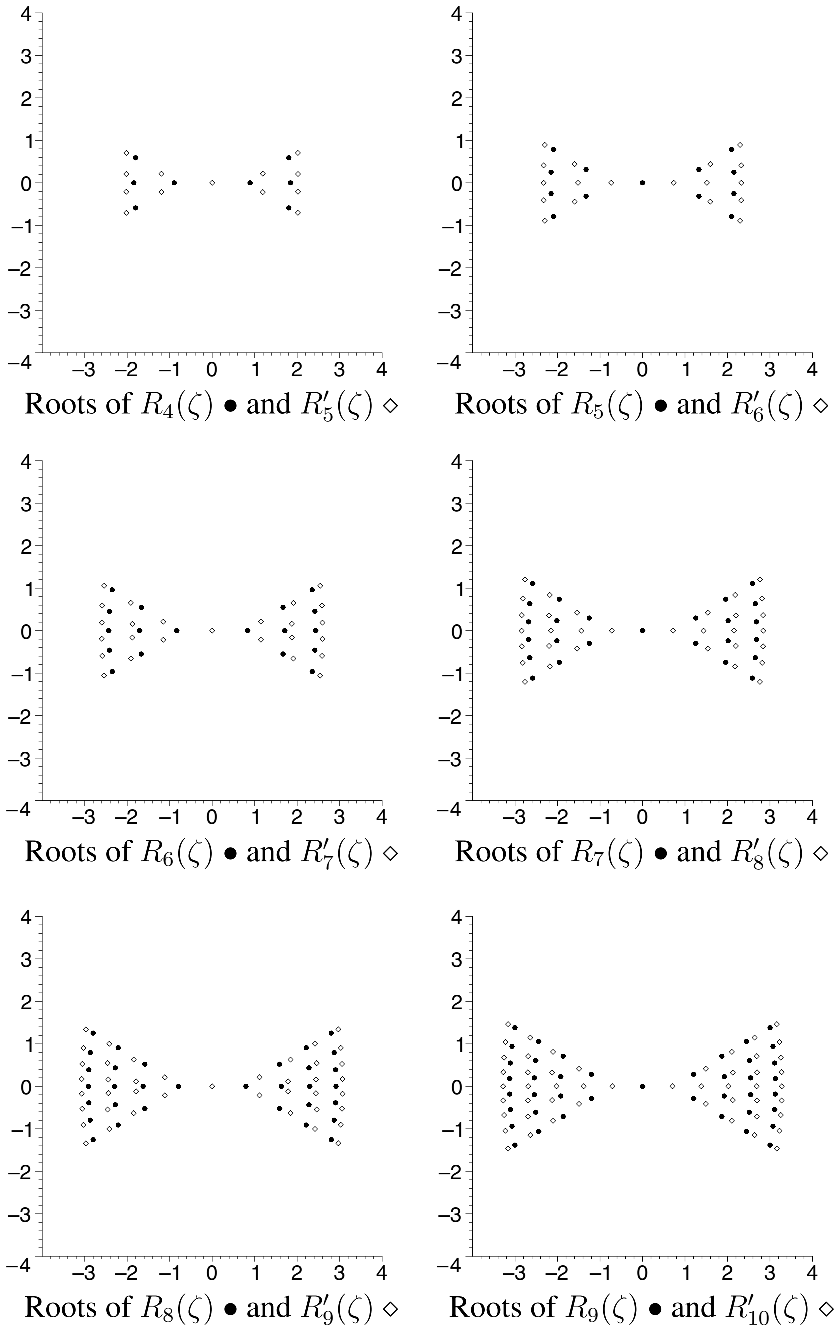

Figure 6. Roots of $R_n(\zeta)$, denoted by •, and $R'_{n+1}(\zeta)$, denoted by ◇, for $n = 4, 5, \ldots, 9$.

with rational solutions of $\mathrm{P_{II}}$. We feel that this 'interlacing of roots' for the polynomials $R_n(\zeta)$ warrants further analytical and numerical studies as does an investigation of the relative locations of the roots for $S_n(z; \mu)$, $S_{n+1}(z; \mu)$ and their derivatives. We shall not pursue these questions any further here.

Further interesting open questions for the polynomials $S_n(z;\mu)$ and $R_n(\zeta)$ are the following.

(i) Do generating functions $\Phi(z,\lambda)$ and $\Psi(\zeta,\lambda)$ for the polynomials $S_n(z;\mu)$ and $R_n(\zeta)$ exist such that

$$\sum_{n=0}^{\infty} S_n(z)\lambda^n = \Phi(z,\lambda) \qquad \sum_{n=0}^{\infty} R_n(\zeta)\lambda^n = \Psi(\zeta,\lambda)?$$

(ii) Do the coefficients of the polynomials $S_n(z;\mu)$ and $R_n(\zeta)$ have combinatorial properties analogous to those for the Yablonskii–Vorob'ev polynomials described in [30, 52, 83]?

Acknowledgments

I wish to thank Mark Ablowitz, Carl Bender, Frédérick Cheyzak, Chris Cosgrove, Andy Hone, Arieh Iserles, Alexander Its, Nalini Joshi, Elizabeth Mansfield and Marta Mazzocco for their helpful comments and illuminating discussions. I also thank the referee for helpful comments and suggestions. This research arose whilst I was involved with the 'Digital Library of Mathematical Functions' project (see http://dlmf.nist.gov); in particular I thank Frank Olver and Dan Lozier for the opportunity to participate in this project. I also thank Colin and Janet Mansfield for their kind hospitality at 'Morning View', Gulgong, Australia, where some of this paper was written.

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